A STUDY OF THE PION-NUCLEUS OPTICAL POTENTIAL

By

Karen Sue Stricker

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Physics

1979

ABSTRACT

A STUDY OF THE PION-NUCLEUS OPTICAL POTENTIAL

By

Karen Sue Stricker

The optical potential model is a convenient means of characterizing the interaction of the pion with the nucleus. Its simplicity makes it practical for the calculation of elastic scattering and pion distorted waves for more complicated processes. Its success in reproducing the early pion data and the existence of new, higher quality data motivate the present investigation.

An optical potential for pion-nucleus interactions in the energy range 0-250 MeV pion kinetic energies is constructed with the Watson multiple scattering series and the πN transition amplitude as starting point. The pion-nucleon to pion-nucleus center of mass transformation is calculated to first order in the ratio of total pion energy to nucleon mass. Multiple scattering corrections in low energy approximation are included to second order in the s-wave and to all orders in the p-wave (the Lorentz-Lorenz or Ericson-Ericson effect). True pion absorption terms, proportional to the square of the nuclear density, are included in both s and p-wave parts of the potential. Pauli blocking is approximated, and an energy shift due to the Coulomb interaction is incorporated.

The potential parameters are taken from the experimental πN phase shifts and theoretical calculations. The potential, of Kisslinger type, is incorporated in coordinate space computer codes which calculate pionic atom level shifts and widths, elastic scattering differential cross sections, and total and partial cross sections. These calculations are compared to the current experimental data. It is found that at low energies (0-50 MeV) the potential produces elastic cross sections which fit the data provided the s-wave repulsion is increased. The pionic atom level data require more absorptive strength than that given by current calculations, as well as increased repulsion consistent with the scattering results. The general features of the resonance region elastic scattering and total cross sections are well reproduced.

ACKNOWLEDGMENTS

My thesis advisor, Professor Hugh McManus, deserves much credit and many thanks for his constant guidance and encouragement throughout this work. I also wish to thank James A. Carr, whose research on a related thesis topic afforded numerous opportunities for productive collaboration, and Professor Dan O. Riska, for many helpful discussions. The various experimental groups whose data are included in this work were most generous in providing their results prior to publication. I am also indebted to several other theorists in the field of pion physics who freely shared their ideas and calculations.

Most of the computations for this thesis were performed on the MSU Cyclotron Laboratory computer, the staff of which gave much valuable assistance. The MSU Physics Department provided time on the main University computer for some of these calculations. I also wish to acknowledge the dedicated, courageous, and expert typing of the rough draft by Mrs. Shari Conroy and the final copy by Mrs. Carol Cole.

Finally, I am pleased to acknowledge my parents, whose support and encouragement were invaluable through my academic life, and my friends at MSU, whose concern and cheerful camaraderie smoothed many rough spots. To Daniel Bauer especially I owe more than I can say for a friendship which has become something much greater.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In the past few years the field of pion-nucleus interactions has advanced rapidly. A large amount of excellent quality data has come out of the intermediate energy laboratories; LAMPF at Los Alamos, New Mexico, USA; TRIUMF in Vancouver, Canada; and SIN in Switzerland. This data includes not only elastic and inelastic differential cross sections, but also total and partial cross sections and single and double charge exchange measurements as well as more complicated reactions. Much progress has also been made in the theoretical description of the pion-nucleus interaction, with characterizations which vary from the phenomenological to the fundamental and microscopic. The present work will focus on the optical potential model, which takes a middle ground between these two approaches and has had a fair amount of success in the description of the early pion-nucleus data.

The concept of an optical potential, that is, a complex potential describing the interaction between the projectile and the nucleus as a whole, and in which the imaginary part accounts for flux lost to other channels from the elastic channel, is due to Bethe (1). An optical model for scattering of high energy particles by nuclei was first introduced by Fernbach, Serber, and Taylor (2) to describe

the scattering of 90 MeV neutrons. They proposed a constant complex potential inside the nucleus, the imaginary part of which can be related to the mean free path λ of the nucleon in nuclear matter (3),

$$Im(U_{opt}) = \frac{k_0}{2M}\frac{1}{\lambda} , \qquad (I\text{-}1)$$

where k_0 is the particle momentum and M its mass. The mean free path can, in turn, be expressed in terms of the total collision cross section and the nuclear density,

$$\frac{1}{\lambda} = \sigma_T \rho . \qquad (I\text{-}2)$$

The optical model was first applied to low energy scattering by Feshbach, Porter, and Weisskopf (4) in the analysis of resonances in 0-3 MeV neutron total cross sections. A theoretical basis for the optical model was provided by Watson (5), who derived the optical potential from a multiple scattering theory. A simplification of the theory, due to the antisymmetry of the target states, was given by Kerman, McManus, and Thaler (6).

The study of the pion-nucleus interaction began with the discovery of the pion in 1947, since early pion experiments usually involved nuclear targets in cloud chambers and emulsions. An optical model for pion-nucleus elastic scattering which included both s- and p-wave terms was first given by Kisslinger (7), and used in the analysis of differential cross section data (8) for 62 MeV π^+ and π^-

on ^{12}C. The analysis of the measured energy shifts and widths of pionic atoms made clear the necessity of including higher order terms in the optical potential. The inclusion of true pion absorption terms, first suggested by Brueckner (9), and the calculation of the Lorentz-Lorenz effect were made by Ericson and Ericson (10) and greatly improved the agreement between calculated and measured levels.

The emphasis shifted to the resonance region with the appearance of π^--^{12}C elastic scattering cross sections at 120, 200, and 280 MeV from CERN (11), followed by data on other nuclei. Although the Kisslinger potential was originally derived for low energy scattering, it was found to give reasonable results in the resonance region also (12). A local optical potential form, the Laplacian model, gave similar results for scattering near resonance (13). Glauber theory (14) was also successfully applied to scattering data in this region (15).

Very little was known of the low energy (0-50 MeV) pion scattering cross sections, and few calculations beyond first order existed for these energies (16) until about 1975, when more accurate data of 50 MeV π^+ elastic scattering from ^{12}C appeared (17). Thies (18) showed that the inclusion of kinematic effects, higher order multiple scattering terms, and s-wave absorption greatly improved the agreement between the calculated and measured cross sections.

Since 1975 the theoretical activity in pion-nucleus interactions has been intense. The most successful microscopic calculations have been the isobar-hole calculations (19) which treat the dominant channel, with Δ_{33} intermediate states, by means of a spreading potential, the parameters of which are fitted to the data. The phenomenological input is small, and the results are encouraging; however, calculations for nuclei larger than ^{16}O are impractical.

The phenomenological optical model has also received a great deal of attention. It has been shown (20) that a first order Kisslinger potential with four free parameters can be fitted to the elastic scattering data for pion kinetic energies around 50 MeV. Four parameters are also sufficient to describe the pionic atom data for a wide range of nuclei (21). The elastic scattering cross sections in the resonance region can also be fit by optical model calculations, requiring, however, a somewhat more sophisticated potential with more than four parameters (22).

The phenomenological approach, although successful in describing various classes of data, has almost no predictive power and is most unsatisfying to a theorist. The microscopic theories have a strong theoretical base and a minimum of approximations but are extremely complicated, tedious calculations and have been made for only a few light nuclei at a few energies. Thus the need at present for a simpler approach based on theoretical considerations but with a simple optical potential form. Such a model, if carefully constructed, should be valid over a fairly wide energy range, say

0-250 MeV, and for all nuclei large enough to justify the optical model assumptions, certainly carbon and all heavier nuclei. The theoretical basis gives the model predictive power; its simplicity makes it a useful tool in calculations of more complicated processes. The important physical content of the theory appears in the optical potential in a straightforward way, not buried in vast computer calculations, giving a feel for the important features of the problem. It is to be hoped that the microscopic calculations will eventually become sufficiently tractable and accurate to be applicable to most nuclei and energies. However, an optical potential type model, taking input from the more sophisticated theories with suitable approximations, will almost certainly be the basis of most practical calculations.

The early work on theoretically based optical potentials by the Ericsons (10) and Thies (18) has already been mentioned. Pieces of the problem have been much discussed by various authors. A review of all such research will not be attempted in this brief introduction; the interested reader is referred to the proceedings of the several recent pion conferences (23). The purpose of this dissertation is to bring together all aspects of the problem in a coherent framework, to construct an optical potential with a broad range of validity. The form chosen for the potential is a coordinate space form of Kisslinger type, local in the sense of depending on only one pion coordinate. Previous investigations suggest that the essential physics survives the approximations necessary to obtain

such a form, which is chosen for its simplicity. The more important test of the validity of the theory is, of course, the accuracy with which it predicts the experimental results, hence the inclusion in this work of calculations of pionic atom level shifts and widths, differential elastic scattering cross sections, and total and partial cross sections for a variety of nuclei and energies.

The reason usually given for the study of the pion-nucleus interaction is the hope that the pion can be used as a probe of nuclear structure once the pion-nucleus dynamics are understood. The nature of the pion, with three isospin states and no spin, and the fact that pions can be absorbed on nucleons, make the pion a unique tool in, for example, the determination of neutron and proton distributions and perhaps the study of correlations between nucleons. There is a growing interest, however, in the pion-nucleus problem itself. The field of intermediate energy physics, of which pion physics is an important part, has become a meeting ground for the nonrelativistic many-body theories of low energy nuclear physics and the relativistic field theories developed in elementary particle physics. Although the present study does not delve deeply into these questions, the development of the potential indicates where these elements enter and provides a base for more detailed calculations. Any improvements to the optical potential model discussed here will almost certainly involve a more careful synthesis of these two aspects of the problem.

The dissertation is divided into six main parts. In the first of these, Chapter II, the information is presented which forms the basis of the optical potential theory: the pion-nucleon interaction, the form of the pion wave equation, and the multiple scattering formalism. From these the first order Kisslinger optical potential is constructed. In Chapter III, the optical potential is refined with the addition of kinematic effects, multiple scattering corrections, true absorption terms, and Pauli and Coulomb effects. This completes the construction of the optical potential; the comparisons to data are discussed in the following three chapters. The first of these, Chapter IV, is a discussion of the optical potential applied to the analysis of pionic atom shifts and widths. Chapter V presents the calculations of elastic differential cross sections, compared to a selection of the available data. In Chapter VI a discussion is given of the calculation of total and partial cross sections, with the results compared to the data. In Chapters V and VI, two different approaches are taken to the choice of parameters for the optical potential. The first is to adopt the parameters as calculated theoretically in the earlier chapters. The second is to extrapolate by simple means the information gained from the pionic atom analysis discussed in Chapter IV to non-zero energies, extending the work of reference 24. The major conclusions of this work are discussed in the final chapter.

The symbols used for some common quantities are given in Table 1. These will be used throughout, unless otherwise noted.

Table 1. Symbols used throughout this work and their interpretation.

Symbol	Meaning
$(\underset{\sim}{k},\omega)$	Momentum and energy of incoming pion
$(\underset{\sim}{k}',\omega')$	Momentum and energy of outgoing pion
$(\underset{\sim}{p},E)$	Momentum and energy of incoming nucleon
$(\underset{\sim}{p}',E')$	Momentum and energy of outgoing nucleon
$(\underset{\sim}{P},E_A)$	Momentum and energy of incoming nucleus
m	Mass of pion
M	Mass of nucleon
M_A	Mass of nucleus
$\underset{\sim}{t}$	Isospin operator for pion
$\underset{\sim}{\tau}$	Isospin operator for nucleon
$\underset{\sim}{\sigma}$	Spin operator for nucleon
cm subscript	Quantities in pion-nucleon center of mass
2cm subscript	Quantities in pion-two nucleon center of mass
no subscript	Quantities in pion-nucleus center of mass (Subscripts are often dropped when only one frame is being considered.)

CHAPTER II

THE FIRST ORDER OPTICAL POTENTIAL

The optical model provides a method by which the pion-nucleus many body problem can be reduced to a one particle equation for the pion, interacting with an optical potential which describes the nucleus. The optical potential can be derived from knowledge of the measured pion-nucleon scattering amplitude and a multiple scattering formalism which relates the pion-nucleon amplitude to the pion-nucleus interaction. In Section 1 of this chapter the pion-nucleon interaction is described. The pion wave equation is discussed in Section 2. The development of the multiple scattering series for the optical potential is given in Section 3, and the first order coordinate space optical potential is given in Section 4.

1. The Pion-Nucleon Interaction

The most prominent feature of pion-nucleon scattering in the energy region 0-300 MeV pion lab kinetic energy is the effect of the πN resonance, denoted Δ_{33}, at 1236 MeV total center of mass energy or about 180 MeV pion kinetic energy. The subscripts 33 refer to the isospin and total angular momentum of the resonance, both of which have the value 3/2. The orbital angular momentum of the state is L = 1. This channel dominates the πN interaction

at these energies, giving rise to a large p-wave term in the scattering amplitude.

The most general scattering amplitude for this problem can be expanded in orbital angular momentum, isospin, and total angular momentum. The s- and p-wave terms of this expansion are

$$f_{\pi N} = (b_0 + b_1 \underset{\sim}{t}\cdot\underset{\sim}{\tau}) + (c_0 + c_1 \underset{\sim}{t}\cdot\underset{\sim}{\tau})\underset{\sim}{k}_{cm}\cdot\underset{\sim}{k}'_{cm}$$

$$+ (s_0 + s_1 \underset{\sim}{t}\cdot\underset{\sim}{\tau})\underset{\sim}{\sigma}\cdot(\underset{\sim}{k}_{cm} \times \underset{\sim}{k}'_{cm}) \ . \qquad \text{(II-1)}$$

The relationships between the coefficients b_i, c_i, and s_i and the measured pion-nucleon phase shifts are derived in Appendix A, and are given by

$$b_0 = \frac{1}{k_{cm}} \frac{1}{3} (2w_3 + w_1)$$

$$b_1 = \frac{1}{k_{cm}} \frac{1}{3} (w_3 - w_1)$$

$$c_0 = \frac{1}{k_{cm}^3} \frac{1}{3} (4w_{33} + 2w_{31} + 2w_{13} + w_{11}) \qquad \text{(II-2)}$$

$$c_1 = \frac{1}{k_{cm}^3} \frac{1}{3} (2w_{33} + w_{31} - 2w_{13} - w_{11})$$

$$s_0 = \frac{1}{k_{cm}^3} \frac{1}{3} (2w_{33} - 2w_{31} + w_{13} - w_{11})$$

$$s_1 = \frac{1}{k_{cm}^3} \frac{1}{3} (w_{33} - w_{31} - w_{13} + w_{11}) \ .$$

The w_i and w_{ij} are related to the $\alpha_{2I,2J}$ of Appendix A by

$$w_i = k_{cm}\alpha^0_{i1}$$
$$w_{ij} = k_{cm}\alpha^1_{ij} \qquad \text{(II-3)}$$

where

$$\alpha^L_{2I,2J} = \frac{\exp(2i\delta^L_{2I,2J}) - 1}{2i\ k_{cm}} \ . \qquad \text{(II-4)}$$

Here I, L, and J are respectively the isospin, orbital angular momentum, and total angular momentum of the system.

The first two terms of equation II-1, referred to as the s- and p-wave terms, are the most important terms for the calculation of pion-nucleus scattering. The third term, also a p-wave term, is usually negligible for pion-nucleus calculations since the nucleon spin is summed over, and will not be discussed further. The d-wave and higher partial waves do not contribute appreciably until energies well above resonance.

The w_i and w_{ij} and the s- and p-wave parameters are shown in Figures 1 and 2, as a function of pion lab kinetic energy T_π. They are computed from the parametrization of the π-nucleon phase shifts given by Rowe, Salomon, and Landau (25), in which an analytic function of energy is fitted to the quantity $k_{cm}^{-(2L+1)}\tan\delta^L_{2I,2J}$ over the energy range 0-400 MeV. The advantage of such a parametrization is that the scattering parameters derived from it vary

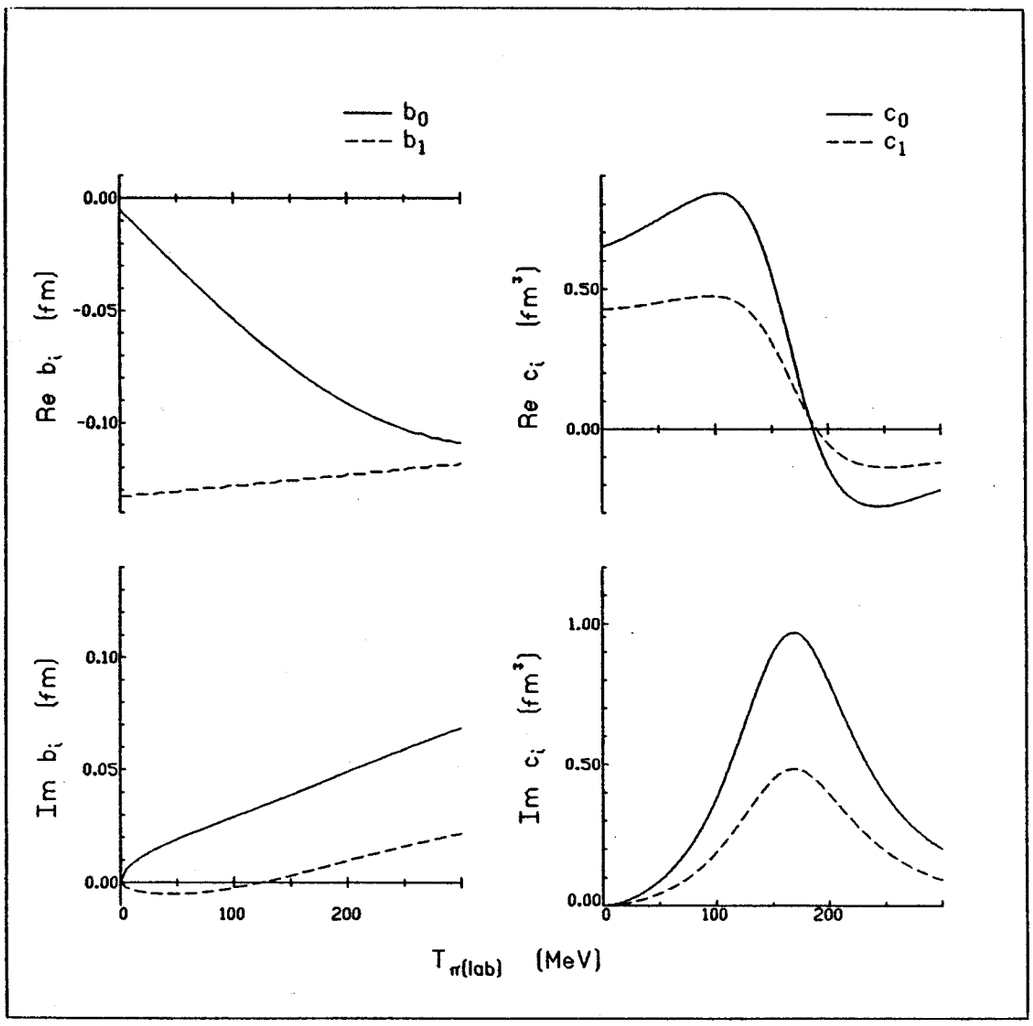

Figure 1. The quantities w_i and w_{ij} as a function of pion lab kinetic energy.

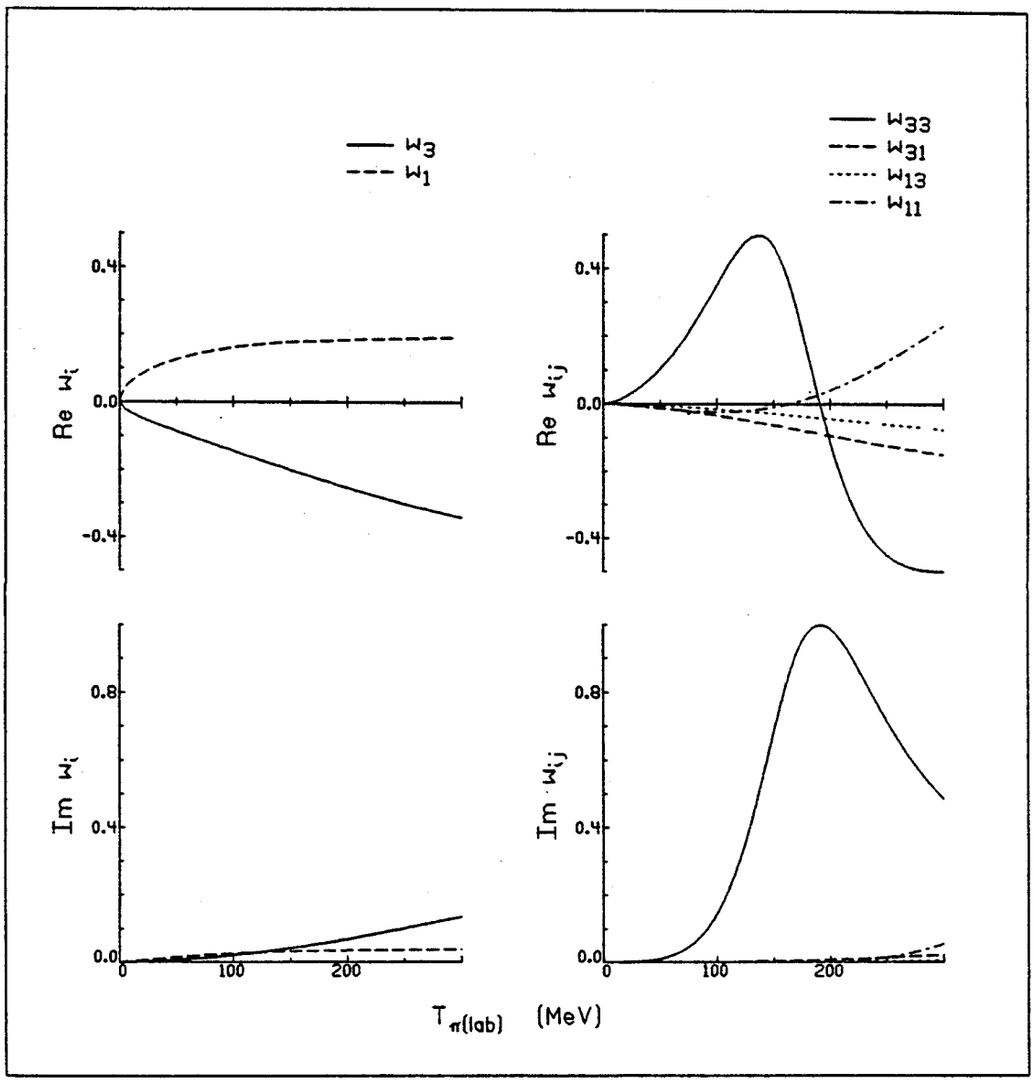

Figure 2. The optical potential parameters b_0, b_1, c_0, and c_1 as a function of pion lab kinetic energy.

smoothly with energy, even at low energies, whereas the scattering parameters calculated directly from phase shifts, even those that have been smoothed, such as the CERN Theoretical set (26), are quite noisy below about 80 MeV.

Several things may be noted. Although the phase shifts $\delta^{L}_{2I,2J}$ are purely real below the threshold for pion production, $\omega_{cm} + E_{cm} = 2m + M$, the scattering parameters are not; the imaginary parts are zero only at zero pion kinetic energy. The real part of the isoscalar s-wave parameter b_0 is negative in this energy region, corresponding to a repulsive s-wave interaction, and is nearly zero at low energies due to a near cancellation of the two terms w_1 and $2w_3$. The p-wave parameters c_0 and c_1 are dominated by the δ^{1}_{33} phase shift and display characteristic resonant behavior, the real part crossing zero at the resonance energy and the imaginary part reaching a maximum at this point. The parameter $\mathrm{Re}(c_0)$ is positive below resonance, hence an attractive p-wave interaction in this region.

The quantity required for the pion-nucleus calculations is the pion-nucleon transition amplitude $t^{\pi N}$. This is related to the pion-nucleon potential v by a Lippmann-Schwinger type equation,

$$t^{\pi N} = v + v2\bar{\omega}_{cm} g^0 t^{\pi N} \ , \qquad \text{(II-5)}$$

where g^0 is the propagator for a free pion; in momentum space

$$g^0 = \frac{1}{-k^2 + k_0^2 + i\varepsilon} \ . \qquad \text{(II-6)}$$

The factor $2\bar{\omega}_{cm}$, where $\bar{\omega}_{cm}$ is the relativistic equivalent of the reduced mass

$$\bar{\omega}_{cm} = \frac{\omega_{cm} E_{cm}}{\omega_{cm} + E_{cm}} \quad , \tag{II-7}$$

is a result of the use of the Klein-Gordon equation rather than the Schrödinger equation for the pion. This point will be considered in more detail in the discussion of the pion-nucleus scattering equation. Matrix elements of $t_{\pi N}$ between momentum states of the pion and nucleon can be written

$$\langle \underset{\sim}{k}', \underset{\sim}{p}' | t^{\pi N} | \underset{\sim}{k}, \underset{\sim}{p} \rangle = (2\pi)^3 \delta(\underset{\sim}{k} + \underset{\sim}{p} - \underset{\sim}{k}' - \underset{\sim}{p}') t^{\pi N}(\underset{\sim}{k}_{cm}, \underset{\sim}{k}'_{cm}) \ . \tag{II-8}$$

The transition amplitude and scattering amplitude can then be shown (27) to be related in the πN center of mass by

$$t^{\pi N}(\underset{\sim}{k}_{cm}, \underset{\sim}{k}'_{cm}) = - \frac{4\pi}{2\bar{\omega}_{cm}} f_{\pi N}(\underset{\sim}{k}_{cm}, \underset{\sim}{k}'_{cm}) \tag{II-9}$$

Thus, the required pion-nucleon T-matrix is related in a simple way to the experimental phase shifts.

2. The Pion Wave Equation

Before discussing the multiple scattering series expansion for the optical potential, it is necessary to consider what form the wave equation for the pion will take. The nucleus is quite

massive and can be treated nonrelativistically. However, the rest mass of the pion is not large compared to its momentum at the energies considered here and must be treated relativistically. Thus, the Hamiltonian for the system must include rest masses and can be written

$$H = (k^2 + m^2)^{\frac{1}{2}} + M_A + \frac{P^2}{2M_A} + V \qquad \text{(II-10)}$$

where $\underset{\sim}{P}$ and M_A are the momentum and mass of the nucleus, and V characterizes the interaction between the nucleons in the target and the pion. The part of the Hamiltonian which describes the internal dynamics of the nucleus has been neglected, assuming that the excitation energies of the various nuclear states do not play an important role. The Schrödinger equation for the system is then

$$[(k^2 + m^2)^{\frac{1}{2}} + M_A + \frac{P^2}{2M_A} + V]\Psi = E_T\Psi \qquad \text{(II-11)}$$

Goldberger and Watson (27) have shown that for $|V| \ll m$ and $\frac{|\nabla V|}{|V|} \ll k_0 \equiv (\omega^2 - m^2)^{\frac{1}{2}}$, this equation is equivalent to a Klein-Gordon-like equation for Ψ,

$$(k^2 + m^2)\Psi = (E_T - V - M_A - \frac{P^2}{2M_A})^2\Psi \qquad \text{(II-12)}$$

In the pion-nucleus center of mass system

$$k^2 = P^2 \qquad \text{(II-13)}$$

and equation II-12 can be written

$$[k^2(1 + \frac{E_T - M_A}{M_A}) + m^2]\Psi = (E_T - M_A - V)^2\Psi \qquad \text{(II-14)}$$

where the terms $(\frac{P^2}{2M_A} V)$ and $(\frac{P^2}{2M_A})^2$ have been dropped. Replacing k^2 by $-\nabla^2$ and rearranging gives

$$\{\nabla^2 + [(E_T - M_A)^2 - m^2] \frac{M_A}{E_T} - 2(E_T - M_A) \frac{M_A}{E_T} V + \frac{M_A}{E_T} V^2\}\Psi = 0 \qquad \text{(II-15)}$$

The quantity $(E_T - M_A)$ can be evaluated by considering the pion far from the nucleus, in which case the total energy of the system is

$$E_T = \omega + M_A + \frac{k_0^2}{2M_A} . \qquad \text{(II-16)}$$

Thus, equation II-15 can be written

$$(\nabla^2 + k_0^2 - 2\bar{\omega} V + \frac{\bar{\omega}}{\omega} V^2)\Psi = 0 \qquad \text{(II-17)}$$

where $\bar{\omega}$ is the reduced energy for the pion-nucleus system,

$$\bar{\omega} = \frac{\omega E_A}{\omega + E_A} . \qquad \text{(II-18)}$$

A similar equation can be written for the bound state case, but with k_0^2 replaced by $-\kappa^2$. It is seen that equation II-17 is very similar

to the usual Schrödinger equation but with (-2mV) replaced by $(-2\bar{\omega}V + \frac{\bar{\omega}}{\omega} V^2)$.

Because the potential V depends on the coordinates of the nucleons as well as the pion, $V = V(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots \underset{\sim}{r}_A; \underset{\sim}{r})$, equation II-17 is still an A + 1 particle equation. The formalism and approximations required to reduce this to a one-body equation for the pion are discussed in the next section. Some simplification can be made at this point by dividing the pion-nucleus potential into two components,

$$V = V_S + V_{EM} \cdot \qquad \text{(II-19)}$$

The potential V_{EM}, describing the electromagnetic interaction between the pion and various nucleons, can be approximated by the Coulomb potential V_C due to a smooth charge distribution corresponding to the measured proton distribution of a given nucleus. With this approximation $V_{EM} = V_C(r)$ depends only on the pion coordinate measured relative to the nuclear center. The strong potential V_S is assumed to be of the form

$$V_S = \sum_i v_i(\underset{\sim}{r}_i, \underset{\sim}{r}) \qquad \text{(II-20)}$$

with $\underset{\sim}{r}_i$ the coordinate of the ith nucleon, $\underset{\sim}{r}$ that of the pion. Note that v_i is just the strong part of the potential v of equation II-5.

The strategy of the next section will be to ignore the Coulomb potential and manipulate the wave equation with potential $V = V_S$ in such a way that the nucleon coordinates can be integrated out. The resulting potential, known as the optical potential, can then be put in a wave equation for the pion that includes the Coulomb potential. This procedure is an approximation, the effect of which is discussed in a later section.

One more approximation is made at this point; the $V_S V_C$ and V_S^2 parts of V^2 are dropped as they are small compared to the ωV_C and ωV_S terms. Then equation II-17 becomes

$$(-k^2 + k_0^2 - 2\bar{\omega}V_S - 2\bar{\omega}V_C + \frac{\bar{\omega}}{\omega} V_C^2)\Psi = 0 \qquad \text{(II-21)}$$

where, in preparation for the calculations of the next section, the momentum space form has been given.

3. The Multiple Scattering Series

The multiple scattering formalism (5) has as its starting point the Lippmann-Schwinger equation (28),

$$T = V + VG^0T \ , \qquad \text{(II-22)}$$

which is equivalent to the Schrödinger equation for a scattering problem. Here the Hamiltonian is given by

$$H = H_0 + V \ , \qquad \text{(II-23)}$$

G^0 is the propagator for the noninteracting pion-nucleus system,

$$G^0 = \frac{1}{E - H_0 + i\varepsilon} , \qquad \text{(II-24)}$$

and T, the transition matrix, is defined by

$$V\psi = T\phi , \qquad \text{(II-25)}$$

where ψ is the wavefunction which is the solution to the full Schrödinger equation,

$$H\psi = E\psi , \qquad \text{(II-26)}$$

and ϕ is the solution to the equation without interaction,

$$H_0\phi = E\phi . \qquad \text{(II-27)}$$

An equation similar to the Lippmann-Schwinger equation can be derived from equation II-17 by defining

$$\hat{V} = 2\bar{\omega}V_s$$

$$\hat{T} = 2\bar{\omega}T \qquad \text{(II-28)}$$

$$\hat{G}^0 = \frac{1}{-k^2 + k_0^2 + i\varepsilon} .$$

Then equation II-17 can be written

$$\hat{T} = \hat{V} + \hat{V}\hat{G}^{0}\hat{T} \qquad \text{(II-29)}$$

and manipulated in the same way as the conventional Lippmann-Schwinger equation. Note that $\hat{T}$, $\hat{V}$, and $\hat{G}$ are (A + 1) particle operators. In this section equation II-29 will be rearranged in order to make use of the knowledge of the pion-nucleon T-matrix and in order to group all the largest terms together in such a way that the nucleon coordinates can be integrated over, reducing the problem to a one particle equation. By defining an optical potential which includes these largest terms and solving the corresponding wave equation, they can be treated exactly.

To derive the optical potential the infinite series implicit in the Lippmann-Schwinger equation,

$$\hat{T} = \hat{V} + \hat{V}\hat{G}^{0}\hat{V} + \hat{V}\hat{G}^{0}\hat{V}\hat{G}^{0}\hat{V} + \ldots \qquad \text{(II-30)}$$

is rearranged in two ways. The derivation given here roughly follows that of Eisenberg (29).

The first rearrangement groups together all interactions which do not have the nuclear ground state as an intermediate state into the subseries

$$\begin{aligned}\hat{U} &= \hat{V} + \hat{V}\hat{G}^{0}(1 - P_0)\hat{V} + \hat{V}\hat{G}^{0}(1 - P_0)\hat{V}\hat{G}^{0}(1 - P_0)\hat{V} + \ldots \\ &= \hat{V} + \hat{V}\hat{G}^{0}(1 - P_0)\hat{U} \qquad \text{(II-31)}\end{aligned}$$

where P_0 is the ground state projection operator $|0><0|$; $(1 - P_0)$ projects onto all other states. It is to be hoped that the ground state expectation value of this series converges rapidly, since the matrix elements of V between the ground state and an excited state are assumed to be much smaller than the ground state to ground state matrix elements. Using the first result of Appendix B, equation II-29 can be written

$$\hat{T} = \hat{U} + \hat{U}\hat{G}^0 P_0 \hat{T} \ . \qquad \text{(II-32)}$$

This series contains all the large terms, i.e. those with ground state intermediate states, and therefore is not expected to converge rapidly. The ground state expectation value of equation II-32 gives

$$\hat{T}_{00} = \hat{U}_{00} + \hat{U}_{00}\hat{G}^0_{00}\hat{T}_{00} \qquad \text{(II-33)}$$

where

$$\hat{U}_{00} = <0|\hat{U}|0> \ , \qquad \text{(II-34)}$$

with similar definitions for $\hat{T}_{00}$ and $\hat{G}^0_{00}$. Note that $\hat{G}^0$ is diagonal in the nuclear states,

$$<n|\hat{G}^0|m> = \hat{G}^0_n \delta_{mn} \ . \qquad \text{(II-35)}$$

Equation II-33 is now a one particle equation for the pion and can be written as a Schrödinger-like equation with U_{00} as potential. Thus the problem can be solved "exactly" (by computer), if U_{00} is known.

The second rearrangement is motivated by the fact that the πN T-matrix, not the potential, is the quantity closely related to the experimental data. The terms in the expansion for $\hat{U}$ which involve only the potential of the ith nucleon can be grouped together to define a quantity similar to the free $t_{\pi N}$ of equation II-5. Writing

$$\hat{U} = \sum_i \hat{U}_i \tag{II-36}$$

and noting that

$$\hat{V} = \sum_i \hat{v}_i \ , \tag{II-37}$$

equation II-31 can be rewritten using the second result of Appendix B,

$$\hat{U} = \sum_i \hat{\tau}_i + \sum_i \sum_{j\neq i} \hat{\tau}_i \hat{G}^0 (1 - P_0) \hat{\tau}_j + \sum_i \sum_{j\neq i} \sum_{k\neq j} \hat{\tau}_i \hat{G}^0 (1 - P_0) \hat{\tau}_j \hat{G}^0 (1 - P_0) \hat{\tau}_k + \ldots \tag{II-38}$$

where τ_i is defined by

$$\hat{\tau}_i = \hat{v}_i + \hat{v}_i \hat{G}^0 (1 - P_0) \hat{\tau}_i \ . \qquad \text{(II-39)}$$

In order to relate $\hat{\tau}_i$ to the free pion-nucleon T-matrix $t^{\pi N}$, define

$$\tau_i = \frac{1}{2\bar{\omega}} \hat{\tau}_i \text{ and } v_i = \frac{1}{2\bar{\omega}} \hat{v}_i \ . \qquad \text{(II-40)}$$

Then

$$\tau_i = v_i + v_i [G^0 (1 - P_0) 2\bar{\omega}] \tau_i \ . \qquad \text{(II-41)}$$

The first result of Appendix B applied to equations II-41 and II-5 gives the relationship of τ_i and the free πN T-matrix for the ith nucleon $t_i^{\pi N}$,

$$\tau_i = t_i^{\pi N} + t_i^{\pi N} [G^0 (1 - P_0) 2\bar{\omega} - g^0 2\bar{\omega}_{cm}] \tau_i \ . \qquad \text{(II-42)}$$

As was pointed out by Kerman, McManus, and Thaler (6), the antisymmetrization of the intermediate states can be exploited in order to simplify equation II-38 for $\hat{U}$. Let $\mathcal{A}$ be a projection operator projecting onto completely antisymmetrized target states. Note that $\mathcal{A}$ commutes with V, P_0, and G^0 since these are totally symmetric in the nucleon coordinates. Thus, assuming T and U will always be taken between properly antisymmetrized states, equations II-29 and II-31 can be written

$$\hat{T} = \hat{V} + \hat{V}\hat{G}^0\mathcal{A}\hat{T} \tag{II-43}$$

$$\hat{U} = \hat{V} + \hat{V}\hat{G}^0(1 - P_0)\mathcal{A}\hat{U} \ . \tag{II-44}$$

Equations II-38, II-39, II-41, and II-42 can be rederived with the operator $\mathcal{A}$ included, yielding equations of the same form but with $\hat{G}^0(1 - P_0)$ replaced by $\hat{G}^0(1 - P_0)\mathcal{A}$. Note that the $\hat{\tau}_i$ thus defined are somewhat different than those defined in equation II-39. With this change the matrix elements of the $\hat{\tau}_i$ in the equivalent of equation II-38 are the same for all i, since with the antisymmetrization all nucleons are equivalent. Equation II-38 becomes

$$\begin{aligned}\hat{U} = A\hat{\tau}_i &+ A(A - 1)\hat{\tau}_i\hat{G}^0(1 - P_0)\mathcal{A}\hat{\tau}_j \\ &+ A(A - 1)^2\hat{\tau}_i\hat{G}^0(1 - P_0)\mathcal{A}\hat{\tau}_j\hat{G}^0(1 - P_0)\mathcal{A}\hat{\tau}_k + \ldots\end{aligned} \tag{II-45}$$

where $i \neq j$, $j \neq k$, and so on. The equation giving τ_i in terms of $t_i^{\pi N}$ is now

$$\tau_i = t_i^{\pi N} + t_i^{\pi N}\,[G^0(1 - P_0)\mathcal{A}2\bar{\omega} - g^0 2\bar{\omega}_{cm}]\,\tau_i \ . \tag{II-46}$$

The difference between τ_i and $t_i^{\pi N}$ will be neglected; this is known as the impulse approximation (30).

The final result, equation II-45, is the multiple scattering series for $\hat{U}$. The first term describes the scattering to all orders from one nucleon, summed over all nucleons. The second term

describes scattering to all orders by one nucleon, propagation, and then scattering to all orders by a second nucleon. The third term describes three such scatterings, and so on. The optical potential is given by

$$2\bar{\omega}U_{opt} = \hat{U}_{00} \equiv \langle 0|\hat{U}|0\rangle \ . \qquad \text{(II-47)}$$

Writing equation II-33 in the form of a Schrödinger equation and including the Coulomb potential gives a wave equation much like equation II-17 but involving pion coordinates only,

$$(\nabla^2 + k_0^2 - 2\bar{\omega}\,(U_{opt} + V_C) + \frac{\bar{\omega}}{\omega}\,V_C^2)\ \phi(r) = 0 \qquad \text{(II-48)}$$

where ϕ is the pion wavefunction.

It is, of course, impossible to calculate all terms of the series for $\hat{U}$, equation II-45. However, the first two terms and a partial summation of the rest can be calculated if some approximations are made. This is the subject of the next chapter.

4. The Optical Potential--Simplest Assumptions

To see the general features of the pion-nucleus optical potential, it is useful to construct the first order potential, arising from the first term in equation II-45. The impulse approximation is made, and kinematic corrections due to the transformation of $t^{\pi N}$ to the pion-nucleus center of mass, and to the difference between ω and $\bar{\omega}$, are ignored. Then

$$2\omega U^{(1)}_{opt} = A\langle 0|2\omega t_i^{\pi N}|0\rangle \qquad \text{(II-49)}$$

which, by equations II-8 and II-9, is

$$2\omega U^{(1)}_{opt}(\underset{\sim}{k},\underset{\sim}{k}') = A\langle 0|(2\pi)^3\delta(\underset{\sim}{k} + \underset{\sim}{p}_1 - \underset{\sim}{k}' - \underset{\sim}{p}_1')(-4\pi)f_{\pi N}(\underset{\sim}{k},\underset{\sim}{k}')|0\rangle \qquad \text{(II-50)}$$

Here $|0\rangle$ represents $\psi_0(\underset{\sim}{p}_1,\underset{\sim}{p}_2\ldots\underset{\sim}{p}_A)$ and $\langle 0|$ represents $\psi_0^*(\underset{\sim}{p}_1',\underset{\sim}{p}_2\ldots\underset{\sim}{p}_A)$. The scattering amplitude $f_{\pi N}$ is given in equation II-1. Since $f_{\pi N}(\underset{\sim}{k},\underset{\sim}{k}')$ is independent of the nucleon momenta in this approximation, equation C-10 of Appendix C can be used to write

$$2\omega U^{(1)}_{opt}(\underset{\sim}{k},\underset{\sim}{k}') = -4\pi A f_{\pi N}(\underset{\sim}{k},\underset{\sim}{k}')\rho(q) \qquad \text{(II-51)}$$

where $\rho(q)$ is the Fourier transform of the nuclear density $\rho(r)$, normalized to 1, and $\underset{\sim}{q}$ is the momentum transfer, $\underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k}$. Assuming a nucleus with N = Z and zero spin, isovector and spin dependent terms can be ignored, and the optical potential becomes

$$2\omega U^{(1)}_{opt}(\underset{\sim}{k},\underset{\sim}{k}') = -4\pi A\{b_0\rho(q) + c_0\rho(q)\underset{\sim}{k}\cdot\underset{\sim}{k}'\} \qquad \text{(II-52)}$$

in momentum space or

$$2\omega U^{(1)}_{opt}(\underset{\sim}{r}) = -4\pi A\{b_0\rho(r) - \underset{\sim}{\nabla}\cdot[c_0\rho(r)]\underset{\sim}{\nabla}\} \qquad \text{(II-53)}$$

in coordinate space. The gradient operators act on all functions of $\underset{\sim}{r}$ to their right (see Appendix C). Equation II-53 is known as the Kisslinger potential (7). This is not the only choice for the form of $U_{opt}^{(1)}$, as will be discussed in the first section of Chapter III, but it is the form adopted in this work.

As might be expected, the Kisslinger potential has s-wave and p-wave parts which are respectively repulsive and attractive for energies below resonance. The imaginary parts of the potential reflect the flux lost from the elastic channel, which process is sometimes called absorption. However, it is important to make a distinction between this process, in which the pion is present in the final state with the nucleus in an excited state, and the process not included in the simple optical potential above, in which the pion is absorbed by the nucleus and does not reappear. The former process will be referred to as quasielastic, and the latter as true absorption. At zero pion kinetic energy there is no energy available for quasielastic processes, and the first order optical potential is purely real.

Due to the p-wave interaction the Kisslinger potential is "non-local" (more accurately, velocity dependent). The p-wave term acting on the pion wavefunction can be written

$$c_0 \underset{\sim}{\nabla}\cdot[\rho(r)\underset{\sim}{\nabla}\phi(r)] = c_0[\underset{\sim}{\nabla}\rho(\underset{\sim}{r})]\cdot[\underset{\sim}{\nabla}\phi(r)] + c_0\rho(r)\nabla^2\phi(r) \ . \qquad \text{(II-54)}$$

The first term contributes a surface peaked interaction. The second term is somewhat troublesome, as it has the same form as the kinetic

energy term but appears with opposite sign below resonance. The difficulty can be seen more clearly when the optical potential, equation II-53, is put in the wave equation II-48,

$$\{[1 - 4\pi c_0\rho(r)]\nabla^2 + k_0^2 + 4\pi b_0\rho(r) - 4\pi c_0[\underset{\sim}{\nabla}\rho(r)]\cdot\underset{\sim}{\nabla}$$

$$- 2\omega V_C + V_C^2\}\phi(r) = 0 \qquad \text{(II-55)}$$

When $\mathrm{Re}(4\pi c_0\rho)$ becomes greater than one, the $\nabla^2\phi$ term changes sign, giving rise to an attractive "potential" in which an infinite number of bound states can exist, with the peculiar property that pion wavefunctions with more nodes correspond to more deeply bound pion states (31). The pion wavefunction can be shown to have a logarithmic singularity at the point where the $\nabla^2\phi$ term changes sign. With $\rho(r) \cong .17$ fm^{-3} in the nuclear interior the requirement $\mathrm{Re}(4\pi c_0\rho(r))>1$ leads to $\mathrm{Re}(c_0) > .47$ fm^3, a condition satisfied by c_0 computed from phase shifts in the entire low energy region. Higher order corrections to be discussed in the next chapter, in particular the Ericson-Ericson effect, reduce the strength of the p-wave term. It is not clear, however, whether this reduction is sufficient to avoid difficulties. It is to be noted that although the pion wavefunction is singular in the interior, its exterior behavior is not anomalous, and the calculated scattering cross sections are perfectly reasonable.

The anomalous behavior of the Kisslinger potential is due, of course, to the approximations made, and in particular to the

off-shell extrapolation chosen. The introduction of form factors, which eliminate the high momentum components, is one possible remedy.

CHAPTER III

THE FULL OPTICAL POTENTIAL

As noted in the introduction, the first order treatment of the optical potential described in the previous chapter was found to be inadequate for the description of pion-nucleus processes, in particular the pionic atom level shifts and widths. This led to studies of kinematic and second order effects in the optical potential. In this chapter the various corrections to the first order optical potential which are incorporated into the calculations are derived.

The first section of this chapter deals with the kinematic transformation which was ignored in the simple optical potential of the last chapter, that is, the transformation of the pion-nucleon T-matrix from the pion-nucleon to the pion-nucleus center of mass. In the second section, the higher order multiple scattering terms are considered, in particular the second order s-wave term and a partial summation of the p-wave terms known as the Ericson-Ericson effect. Terms which arise from true absorption are discussed in Section 3. Other corrections, due to the Pauli exclusion principle and Coulomb distortion, are described in Section 4. Finally, the full optical potential is stated in Section 5.

In order to give some indication of the effect of the various kinematic and higher order corrections in pion-nucleus calculations, a representative set of differential elastic scattering calculations is shown where appropriate. Calculations for the nuclei ^{16}O and ^{208}Pb at 50 and 162 MeV are given to illustrate the nucleon number and energy dependence of the effects. Only π^+ scattering is shown in most cases, since the π^- scattering shows similar changes. The low energy data shown is that of Ref. 32 (diamonds), and Refs. 33 and 34 (triangles). The data at 162 MeV is from Ref. 35 (^{16}O) and Ref. 36 (^{208}Pb).

1. Kinematics

The pion-nucleon transition matrix $\tau_i \cong t_i^{\pi N}$ which is required for the optical potential is simply related to the experimentally measured scattering amplitude $f_{\pi N}$, defined in the πN center of mass, where $|\underset{\sim}{k}| = |\underset{\sim}{k}'|$. However, τ_i must be known for $\underset{\sim}{k} + \underset{\sim}{p} \neq 0$ as well. If one ignores the Fermi motion of the nucleons within the nucleus τ_i must be calculated in the pion-nucleus center of mass. This is sometimes referred to as the angle transformation, since it involves the transformation of the angle between $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ in the p-wave term, among other things. When Fermi motion is included the transformation depends on $\underset{\sim}{p}$ as well as $\underset{\sim}{k}$. It is a straightforward matter to relate $t^{\pi N}$ for $\underset{\sim}{k} + \underset{\sim}{p} = \underset{\sim}{Q}$ to the πN center of mass amplitude for which $\underset{\sim}{k} + \underset{\sim}{p} = 0$. If, however, $|\underset{\sim}{k}| \neq |\underset{\sim}{k}'|$ in the frame in which $\underset{\sim}{k} + \underset{\sim}{p} = 0$, some assumption must be made about the off energy shell behavior of $t^{\pi N}$. An infinite number of such assumptions exist.

A complete theory of the pion-nucleon system would provide a unique off-shell extrapolation; however, such a theory does not yet exist.

The Kisslinger potential assumes that $t^{\pi N}$ is proportional to $b_0 + c_0 \underset{\sim}{k}\cdot\underset{\sim}{k}'$ for all $\underset{\sim}{k}$ and $\underset{\sim}{k}'$. A different off shell amplitude can be obtained if the scattering amplitude

$$f_{\pi N} = b_0 + c_0 k\cdot k' \qquad \text{(III-1)}$$

is rewritten using

$$\underset{\sim}{k}\cdot\underset{\sim}{k}' = \tfrac{1}{2}(k^2 - q^2) \ , \qquad \text{(III-2)}$$

which gives

$$f_{\pi N} = b_0' + c_0' q^2 \qquad \text{(III-3)}$$

where $b_0' = b_0 + \frac{1}{2} k^2 c_0$, $c_0' = -\frac{1}{2} c_0$, and $\underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k}$. This leads to an optical potential, with the simplifying assumptions of the previous chapter,

$$2\omega U_{opt}(r) = -4\pi\{b_0'\rho(r) + c_0'[\nabla^2\rho(r)]\} \ , \qquad \text{(III-4)}$$

which is generally called the Laplacian model. Note that the ∇^2 acts only on $\rho(r)$, making this a local potential. The Equation III-2

is only true on shell, therefore the off shell behavior of these two potentials is somewhat different.

Figure 3 shows a comparison of differential cross sections calculated with these two potentials. The differences are pronounced, especially at 50 MeV, where the two curves are of quite different character. At 162 MeV the curves have different magnitude but more or less the same shape. The partial cross sections also show large differences at 50 MeV. For ^{16}O the reaction cross section calculated with the Kisslinger potential is three times that for the Laplacian potential, with a corresponding inequality in the total cross sections; the elastic cross sections are about equal. For ^{208}Pb the Kisslinger reaction cross section is also greater than the Laplacian; however, the total elastic scattering cross section for the Kisslinger potential is only about half that of the Laplacian, leading to a smaller total cross section. At 162 MeV the cross sections are much more similar; those of the Laplacian potential are slightly larger.

The Kisslinger and Laplacian potentials have been the most popular models for pion-nucleus scattering and are easily transformed to coordinate space. Another type of potential is known as the separable potential because the $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ dependence of the pion-nucleon potential v is assumed to be of the form $\pm g(\underset{\sim}{k})g(\underset{\sim}{k}')$ in each channel, leading to a $t^{\pi N}$ of the form $\pm g(\underset{\sim}{k})g(\underset{\sim}{k}')D(E)$ in each channel. The g's, known as form factors, reflect the finite range of the πN interaction. This assumed form is quite useful in that the form factors and D can be related to the πN phase shifts (37) and give

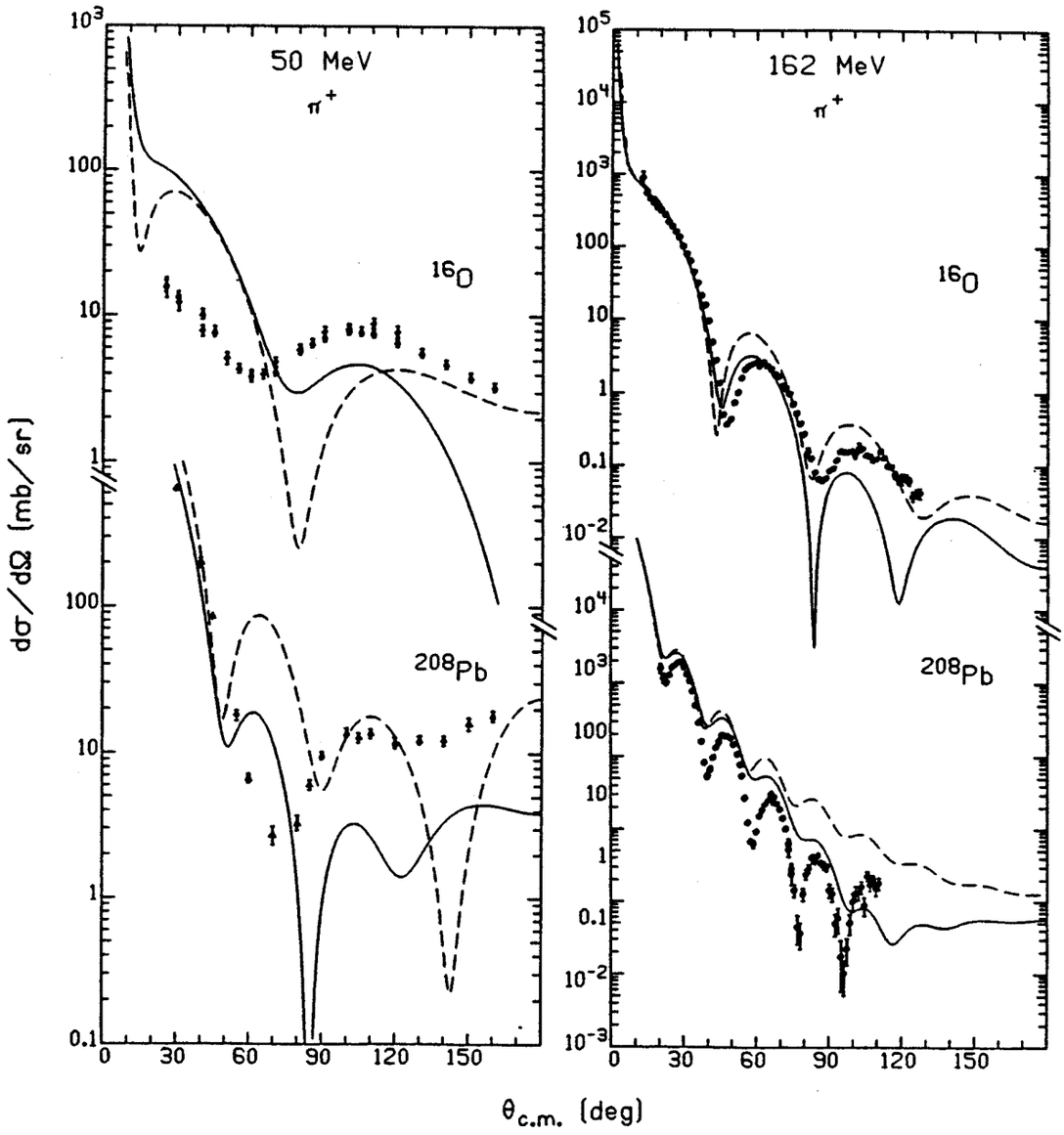

Figure 3. Comparison of calculations with the Kisslinger (solid curve) and Laplacian (dashed curve) optical potentials.

a more realistic off shell behavior for $t^{\pi N}$; however, the coordinate space potential derived from such a theory is of an awkward non-local form.

The Kisslinger form will be adopted for the optical potential in this work. Because it is explicitly separated into s- and p-wave parts it is best for pionic atom analysis and low energy scattering. The form is convenient, also, for the calculation of higher order terms. In the low energy region the Kisslinger potential parameters vary slowly with energy. This is not true in the resonance region, however. It has been shown that when higher order multiple scattering terms are included, taking account of the correlations between nucleons, Kisslinger and Laplacian potentials give similar results (30). This is true since the interaction is of short range and the correlations insure that nucleons are not close together, so that the potentials due to different nucleons are almost non-overlapping. Thus Beg's theorem (38) is applicable, which states that for non-overlapping potentials the scattering depends only on the on-shell part of the potential.

Once the off-shell behavior of the pion-nucleon T-matrix is chosen, all necessary T-matrix elements can be calculated. One method of doing this relativistically is given by relativistic potential theory (39). The process is more complicated than a Lorentz transformation, since the T-matrix does not have well-defined transformation properties (see Appendix D). The relativistic potential theory provides a prescription (40) for relating $\langle \underset{\sim}{k}', \underset{\sim}{p}' | t(E_T) | \underset{\sim}{k}, \underset{\sim}{p} \rangle$,

where E_T is the total energy of the pion-nucleon system, to $t(w,\underset{\sim}{q}',\underset{\sim}{q})$, where w is an energy parameter related to E_T. The momentum $\underset{\sim}{q}$ is related to $\underset{\sim}{k}$ and $\underset{\sim}{p}$, and $\underset{\sim}{q}'$ is related to $\underset{\sim}{k}'$ and $\underset{\sim}{p}'$, by equation D-2 of Appendix D, which is equivalent to a Lorentz transformation to the two-particle center of mass frame. Thus $\underset{\sim}{q} = \underset{\sim}{k}_{cm}$ and $\underset{\sim}{q}' = \underset{\sim}{k}'_{cm}$. It is to be noted that the β for the transformation $(\underset{\sim}{k},\underset{\sim}{p}) \rightarrow (\underset{\sim}{q},-\underset{\sim}{q})$ is not in general the same as the β for $(\underset{\sim}{k}',\underset{\sim}{p}') \rightarrow (\underset{\sim}{q}',-\underset{\sim}{q}')$. The quantity $t(w,\underset{\sim}{q},\underset{\sim}{q}')$ with $|\underset{\sim}{q}| = |\underset{\sim}{q}'|$ is just the on-shell T-matrix and with $|\underset{\sim}{q}| \neq |\underset{\sim}{q}'|$ is its off-shell extrapolation.

The exact expression for $\langle \underset{\sim}{k}',\underset{\sim}{p}'|t(E_T)|\underset{\sim}{k},\underset{\sim}{p}\rangle$ in terms of $t(w,\underset{\sim}{q},\underset{\sim}{q}')$ is given by equations D-5, D-6, and D-10, along with an expansion of the result in powers of Q^2, where $\underset{\sim}{Q} = \underset{\sim}{k} + \underset{\sim}{p} = \underset{\sim}{k}' + \underset{\sim}{p}'$, given by equation D-11. Keeping only the first term in the expansion,

$$\langle \underset{\sim}{k}',\underset{\sim}{p}'|t(E_T)|\underset{\sim}{k},\underset{\sim}{p}\rangle = (2\pi)^3\delta(\underset{\sim}{k}' + \underset{\sim}{p}' - \underset{\sim}{k} - \underset{\sim}{p})Nt(w,\underset{\sim}{k}'_{cm},\underset{\sim}{k}_{cm}) \qquad \text{(III-5)}$$

where

$$N = \left[\frac{E\omega E'\omega'}{E_{cm}\omega_{cm}E'_{cm}\omega'_{cm}} \frac{(E_{cm} + \omega_{cm})(E'_{cm} + \omega'_{cm})}{(E + \omega)(E' + \omega')}\right]^{-\frac{1}{2}} \qquad \text{(III-6)}$$

and

$$w = (E_T^2 - Q^2)^{\frac{1}{2}} \qquad \text{(III-7)}$$

Note that

$$E = (p^2 + M^2)^{\frac{1}{2}} \qquad E_{cm} = (k_{cm}^2 + M^2)^{\frac{1}{2}}$$

$$\omega = (k^2 + m^2)^{\frac{1}{2}} \qquad \omega_{cm} = (k_{cm}^2 + m^2)^{\frac{1}{2}} \tag{III-8}$$

are not the same as the on-shell values usually represented by these symbols.

The off-shell forms discussed previously were given for $f_{\pi N}$. Since $t^{\pi N}$ is related to $f_{\pi N}$ by

$$t^{\pi N}(w,\underset{\sim}{k}'_{cm},\underset{\sim}{k}_{cm}) = -\frac{4\pi}{2\bar{\omega}_{cm}} f_{\pi N}(w,\underset{\sim}{k}'_{cm},\underset{\sim}{k}_{cm}) \tag{III-9}$$

where

$$\bar{\omega}_{cm} = \frac{\omega_{cm}E_{cm}}{\omega_{cm} + E_{cm}} ,$$

an off-shell form is needed for the reduced energy $\bar{\omega}_{cm}$. This is chosen to be symmetric in incoming and outgoing particle energies,

$$\bar{\omega}_{cm} = \left[\frac{\omega_{cm}E_{cm}}{\omega_{cm} + E_{cm}} \frac{\omega'_{cm}E'_{cm}}{\omega'_{cm} + E'_{cm}}\right]^{\frac{1}{2}} . \tag{III-10}$$

Taking the impulse approximation

$$\tau_i = t_i^{\pi N} \tag{III-11}$$

and recalling

$$\hat{\tau}_i = 2\bar{\omega}\tau_i \ , \tag{III-12}$$

it is seen that off-shell matrix elements of $\hat{\tau}_i$ also require an off-shell form for $\bar{\omega}$, taken as

$$\bar{\omega} = \left[\frac{\omega E_A}{\omega + E_A} \frac{\omega' E'_A}{\omega' + E'_A}\right]^{\frac{1}{2}} \tag{III-13}$$

with E_A and E'_A the initial and final energies of the nucleus. Putting all these factors together gives

$$\begin{aligned} &\langle \underset{\sim}{k}', \underset{\sim}{p}'_i | \tau_i(E_T) | \underset{\sim}{k}, \underset{\sim}{p}_i \rangle = -4\pi(2\pi)^3 \\ &\times \delta(\underset{\sim}{k}' + \underset{\sim}{p}'_i - \underset{\sim}{k} - \underset{\sim}{p}_i) \frac{\bar{\omega}}{\omega_{cm}} N f_{\pi N}(w, \underset{\sim}{k}'_{cm}, \underset{\sim}{k}_{cm}) \end{aligned} \tag{III-14}$$

In order to simplify this result, the nucleons and nucleus are assumed nonrelativistic, so that

$$E = E' = E_{cm} = E'_{cm} = M \quad \text{and} \quad E_A = E'_A = M_A = AM \ . \tag{III-15}$$

This is not sufficient to make equation III-14 usable in a coordinate space calculation, as ω and ω' still depend on $\underset{\sim}{k}$ and $\underset{\sim}{k}'$. Therefore, ω and ω' are set equal to their on-shell values

$$\omega = \omega' = \omega_0 \equiv (k_0^2 + m^2)^{\frac{1}{2}} . \tag{III-16}$$

With these approximations equation III-14 becomes

$$\langle \underset{\sim}{k}', \underset{\sim}{p}_i' | \tau_i(E_T) | \underset{\sim}{k}, \underset{\sim}{p}_i \rangle = -4\pi(2\pi)^3$$

$$\times \delta(\underset{\sim}{k}' + \underset{\sim}{p}_i' - \underset{\sim}{k} - \underset{\sim}{p}_i) \frac{1 + \varepsilon}{1 + \varepsilon/A} f_{\pi N}(w, \underset{\sim}{k}'_{cm}, \underset{\sim}{k}_{cm}) \qquad \text{(III-17)}$$

where

$$\varepsilon = \frac{\omega_0}{M} \qquad \text{(III-18)}$$

and the difference between $\frac{\omega_0}{M}$ and $\frac{\omega_{0cm}}{M}$, of order ε^2, has been neglected. At 50 MeV ε = .20; at 200 MeV ε = .36.

The arguments of $f_{\pi N}$, $\underset{\sim}{k}_{cm}$ and $\underset{\sim}{k}'_{cm}$, must be expressed in terms of $\underset{\sim}{k}$, $\underset{\sim}{p}$, $\underset{\sim}{k}'$, and $\underset{\sim}{p}'$. For this equation D-2 or equivalently the Lorentz transformation can be used. The latter gives

$$\underset{\sim}{k}_{cm} = \underset{\sim}{k} + \underset{\sim}{\beta}\gamma\left(\frac{\gamma}{\gamma + 1} \underset{\sim}{\beta} \cdot \underset{\sim}{k} - \omega\right)$$

$$\underset{\sim}{k}'_{cm} = \underset{\sim}{k}' + \underset{\sim}{\beta}'\gamma'\left(\frac{\gamma'}{\gamma' + 1} \underset{\sim}{\beta}' \cdot \underset{\sim}{k}' - \omega'\right) \qquad \text{(III-19)}$$

where

$$\underset{\sim}{\beta} = \frac{\underset{\sim}{k} + \underset{\sim}{p}}{E + \omega} \quad \text{and} \quad \underset{\sim}{\beta}' = \frac{\underset{\sim}{k}' + \underset{\sim}{p}'}{E' + \omega'} \qquad \text{(III-20)}$$

To make these expressions tractable, only terms of first order in $\underset{\sim}{\beta}$ will be kept. (This is nearly equivalent to expanding to first order in ε.) At T_π = 200 MeV $\langle\beta\rangle \cong .25$ and the error due to the

dropped terms is about 6%. With the additional approximations given in III-15 and III-16, equation III-19 becomes

$$\underset{\sim}{k}_{cm} = \frac{M\underset{\sim}{k} - \omega_0\underset{\sim}{p}}{M + \omega_0} = \frac{\underset{\sim}{k} - \varepsilon\underset{\sim}{p}}{1 + \varepsilon}$$

$$\underset{\sim}{k}'_{cm} = \frac{M\underset{\sim}{k}' - \omega_0\underset{\sim}{p}'}{M + \omega_0} = \frac{\underset{\sim}{k}' - \varepsilon\underset{\sim}{p}'}{1 + \varepsilon} \quad , \tag{III-21}$$

and the p-wave term in $f_{\pi N}$ is

$$\underset{\sim}{k}_{cm}\cdot\underset{\sim}{k}'_{cm} = \frac{1}{(1 + \varepsilon)^2}\left[\underset{\sim}{k}\cdot\underset{\sim}{k}' - \varepsilon(\underset{\sim}{k}\cdot\underset{\sim}{p}' + \underset{\sim}{k}'\cdot\underset{\sim}{p}) + \varepsilon^2\underset{\sim}{p}\cdot\underset{\sim}{p}'\right] . \tag{III-22}$$

The last term is of order ε^2 and should be dropped. However, it is an induced s-wave term and, as noted by Brown, Jennings, and Rostokin (41), is important since the first order s-wave term is unusually small. A more careful calculation, too tedious to be given here, indicates that this is the only important ε^2 term.

Because the nucleon is part of a moving nucleus, the nucleon momentum should be separated into a part due to the momentum of the nucleus as a whole and a part due to the momentum of the nucleon relative to the nucleus,

$$\underset{\sim}{p} = -\frac{1}{A}\underset{\sim}{k} + \underset{\sim}{p}_0 \tag{III-23}$$

However, this separation complicates greatly the process of transforming from momentum space to coordinate space and only contributes

terms of order ε/A which are negligibly small for all but the lightest nuclei. Therefore $\underset{\sim}{p}$ and $\underset{\sim}{p}_0$ will be considered equivalent.

Let

$$\underset{\sim}{P} = \frac{\underset{\sim}{p} + \underset{\sim}{p}'}{2} \qquad \underset{\sim}{K} = \frac{\underset{\sim}{k} + \underset{\sim}{k}'}{2}$$

$$\underset{\sim}{\overline{P}} = \underset{\sim}{p}' - \underset{\sim}{p} \qquad \underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k} \tag{III-24}$$

Then equation III-22 can be written

$$\underset{\sim}{k}_{cm} \cdot \underset{\sim}{k}'_{cm} = \frac{1}{(1+\varepsilon)^2} \underset{\sim}{k} \cdot \underset{\sim}{k}' - \frac{2}{1+\varepsilon} \underset{\sim}{P} \cdot \underset{\sim}{K}$$

$$- \frac{1}{2} \frac{1}{1+\varepsilon} q^2 + \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 \underset{\sim}{p} \cdot \underset{\sim}{p}' \tag{III-25}$$

The T-matrix is now expressed in terms of pion-nucleus center of mass quantities, and the first order term of the optical potential,

$$2\bar{\omega} U^{(1)}_{opt} = A\langle 0 | \tau_i | 0 \rangle \tag{III-26}$$

can be calculated. Omitting the spin term this is

$$2\bar{\omega} U^{(1)}_{opt} = -4\pi A \langle 0 | (2\pi)^3 \delta(\underset{\sim}{p}_i + \underset{\sim}{k} - \underset{\sim}{p}'_i - \underset{\sim}{k}') \{(1+\varepsilon)(b_0 + b_1 \underset{\sim}{t} \cdot \underset{\sim}{\tau}_i)$$

$$+ (c_0 + c_1 \underset{\sim}{t} \cdot \underset{\sim}{\tau}_i)\left[\frac{1}{1+\varepsilon} \underset{\sim}{k} \cdot \underset{\sim}{k}' - 2\underset{\sim}{P} \cdot \underset{\sim}{K} - \frac{1}{2} q^2 + \frac{\varepsilon^2}{1+\varepsilon} \underset{\sim}{p} \cdot \underset{\sim}{p}'\right]\} | 0 \rangle \tag{III-27}$$

where it is assumed that $\underset{\sim}{p}'_j = \underset{\sim}{p}_j$ for $j \neq i$. The integrations over ground state nucleon momenta of the various terms are given in Appendix C. (This process is called Fermi averaging.) The result is

$$2\bar{\omega}U^{(1)}_{opt}(\underset{\sim}{k},\underset{\sim}{k}') = -4\pi\{p_1[b_0\rho(q) + \varepsilon_\pi b_1(\rho_p(q) - \rho_n(q))]$$

$$+ p_1^{-1}[c_0\rho(q) + \varepsilon_\pi c_1(\rho_p(q) - \rho_n(q))]\underset{\sim}{k}\cdot\underset{\sim}{k}'$$

$$- \frac{1}{2}(1 - p_1^{-1})[c_0\rho(q) + \varepsilon_\pi c_1(\rho_p(q) - \rho_n(q))]q^2$$

$$+ \frac{(p_1 - 1)^2}{p_1} c_0\kappa(q)\} \qquad \text{(III-28)}$$

where ε_π is the pion charge, ± 1, $p_1 = 1 + \varepsilon$, and $\kappa(q)$ is the Fourier transform of 2M times the kinetic energy density of the nucleus, given in equation C-13. The transformation to coordinate space gives

$$2\bar{\omega}U^{(1)}_{opt}(\underset{\sim}{r}) = -4\pi\{p_1[b_0\rho(r) + \varepsilon_\pi b_1(\rho_p(r) - \rho_n(r))]$$

$$- p_1^{-1}\underset{\sim}{\nabla}\cdot[c_0\rho(r) + \varepsilon_\pi c_1(\rho_p(r) - \rho_n(r))]\underset{\sim}{\nabla}$$

$$+ \frac{1}{2}(1 - p_1^{-1})\nabla^2[c_0\rho(r) + \varepsilon_\pi c_1(\rho_p(r) - \rho_n(r))]$$

$$+ \frac{(p_1 - 1)^2}{p_1} c_0\kappa(r)\} \; . \qquad \text{(III-29)}$$

The nucleon, neutron, and proton densities ρ, ρ_n, and ρ_p are normalized to A, N, and Z, respectively.

This method of treating $\underset{\sim}{k}_{cm}\cdot\underset{\sim}{k}'_{cm}$ is, of course, not unique. One could, for example, replace q^2 by $2k_0^2 - 2\underset{\sim}{k}\cdot\underset{\sim}{k}'$, leading to the optical potential

$$2\bar{\omega}U_{opt}^{(1)}(r) = -4\pi\{p_1 b_0\rho(r) - c_0\underset{\sim}{\nabla}\cdot\rho(r)\underset{\sim}{\nabla} - k_0^2(1 - p_1^{-1})c_0\rho(r) + \frac{(p_1 - 1)^2}{p_1}c_0\kappa(r)\} \qquad \text{(III-30)}$$

where isovector terms have been suppressed for simplicity. It should be noted that any choice of kinematics must treat $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ symmetrically; otherwise the potential is not Hermitian and the results may violate unitarity.

The energy parameter w, given by equation III-7, is the energy at which the scattering parameters should be evaluated and should also be expressed in terms of $\underset{\sim}{p}_i$ and $\underset{\sim}{p}'_i$ before the integrals over nucleon momenta are performed. This is not practicable, however, and w^2 is evaluated with E_T and Q the total energy and momentum of the nucleon and pion before collision,

$$w^2 = (E + \omega)^2 - (\underset{\sim}{p} + \underset{\sim}{k})^2 \ . \qquad \text{(III-31)}$$

This is just the total energy in the pion-nucleon center of mass.